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ON THE MOTION OF A RIGID STAMP ALONG THE BOUNDARY OF AN ORTHOTROPIC VISCOELASTIC HALF-PLANE*

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There are investigated the contact stresses under a rigid stamp moving at a constant velocity along the boundary of an orthotropic viscoelastic half-plane. The problem reduces to solving a singular integral equation whose kernel is represented in the form of the sum of Cauchy-type singularities and a certain regular function. By using Chebyshev polynomials, a solution is constructed for this integral equation. The particular case when only shear creep holds is examined as an illustration.

1. Formulation of the problem. We consider the plane problem of motion of a stamp on which a force acts that is the resultant of the pressure existing at the contact area. Let the stamp move over the boundary of the half-plane at a certain given constant velocity w. We shall consider there to be no friction forces between the stamp and the viscoelastic body, and the dimensions of the contact area are given. In the case of arbitrary viscoelastic anisotropy, the problem can be solved by the same method, however quite awkward expressions result, consequently we limit ourselves to the consideration of an orthotropic viscoelastic body.

The equations relating the strain and stress components in a moving coordinate system will be /1/

$$\varepsilon_x = E^{-1}\sigma_x + \int_{-\infty}^{x} \sigma_x K_{11}(x-\xi)\,d\xi - \nu E^{-1}\sigma_y - \int_{-\infty}^{x} \sigma_y K_{12}(x-\xi)\,d\xi \tag{1.1}$$

$$\varepsilon_y = -\nu E^{-1}\sigma_x - \int_{-\infty}^{x} \sigma_x K_{12}(x-\xi)\,d\xi + E^{-1}\sigma_y + \int_{-\infty}^{x} \sigma_y K_{22}(x-\xi)\,d\xi$$

$$\gamma_{xy} = 2(1+\nu)E^{-1}\tau_{xy} + \int_{-\infty}^{x} \tau_{xy} K_{33}(x-\xi)\,d\xi$$

$$K_{ij}(x-\xi) = K_{ij}{}^{\circ}[w(t-\tau)] \qquad (i, j = 1, 2, 3)$$

To obtain the complete system of equations, the equilibrium equations and relationships connecting the strain components to the displacements /1/ must also be written down (we neglect inertial forces in the case under consideration).

Because there are no friction forces between the stamp and the viscoelastic body, the tangential forces on the boundary of the viscoelastic half-plane are zero: $\tau_{xy} = 0$. We shall assume the surface of the body outside the stamp to be force-free $\sigma_y = 0$. We consider that $\sigma_y =$ const on the section under the stamp.

2. Finding the Green's function. The equilibrium conditions will be satisfied if the stress function $\varphi(x, y)$ is introduced. Taking into account (1.1), and the compatibility condition, and performing a number of manipulations, we obtain

$$E^{-1}\Delta^2\varphi + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\int_{-\infty}^{x} \frac{\partial^2\varphi}{\partial\xi^2} K_{12}(x-\xi)\,d\xi + \tag{2.1}$$

$$\frac{\partial^2}{\partial y^2}\int_{-\infty}^{x} \frac{\partial^2\varphi}{\partial y^2} K_{11}(x-\xi)\,d\xi - \frac{\partial^2}{\partial x^2}\int_{-\infty}^{x} \frac{\partial^2\varphi}{\partial y^2} K_{22}(x-\xi)\,d\xi +$$

$$\frac{\partial^2}{\partial x\partial y}\int_{-\infty}^{x} \frac{\partial^2\varphi}{\partial\xi\partial y} K_{33}(x-\xi)\,d\xi = 0$$

$$\left(\sigma_x = \frac{\partial^2\varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y}\right)$$

We seek the solution of (2.1) in the form

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$$\varphi(x, y) = \exp(i\beta x)\,\psi(y) \tag{2.2}$$

We obtain an ordinary differential equation with constant coefficients in the function $\psi(y)$

$$\begin{gathered} A\psi^{IV}(y) + B\psi''(y) + C\psi(y) = 0 \\ A = E^{-1} + K_{11}^*(i\beta), \quad C = \beta^4[E^{-1} + K_{12}^*(i\beta)] \\ B = \beta^2[-2E^{-1} + K_{12}^*(i\beta) + K_{22}^*(i\beta) - K_{33}^*(i\beta)] \\ K_{lj}^*(i\beta) = \int_0^\infty K_{lj}(S)\exp(-i\beta S)\,dS \qquad (l, j = 1, 2, 3) \end{gathered} \tag{2.3}$$

($S = x - \xi$ is a new independent variable).

All the roots of the corresponding characteristic equation are distinct, hence the solution of (2.3) has the form

$$\psi(y) = \sum_{j=1}^{4} C_j \exp(k_j y), \qquad k_j = \pm\left[\frac{-B \pm (B^2 - 4AC)^{1/2}}{2A}\right]^{1/2} \tag{2.4}$$

Using the boundary conditions, we determine the coefficients

$$C_1 = C_2 = 0, \quad C_3 = k_4/(k_4 - k_3), \quad C_4 = k_3/(k_3 - k_4) \tag{2.5}$$

and we furthermore obtain

$$\begin{gathered} k_3 = B_-\exp(-{}^1/_2 iA_{1-}A_{2-}^{-1}), \quad k_4 = -B_+\exp(-{}^1/_2 iA_{1+}A_{2+}^{-1}) \\ B_\pm = (A_{1\pm}^2 + A_{2\pm}^2)^{1/4}, \quad A_{1\pm} = F\sin\eta \pm D\sin\alpha \\ A_{2\pm} = F\cos\eta \pm D\cos\alpha \\ D = \frac{\beta^2}{2\omega}\{[R_1^*(\beta)]^2 + [R_1^{**}(\beta)]^2\}^{1/4} \\ F = \frac{\beta^2}{2\omega}\{[-2E^{-1} + R_2^*(\beta)]^2 + [R_2^{**}(\beta)]^2\}^{1/4} \\ \omega = \{[E^{-1} + R_{11}^*(\beta)]^2 + [R_{11}^{**}(\beta)]^2\}^{1/2} \\ \alpha = \frac{R_1^{**}(\beta)}{R_1^*(\beta)} - \frac{R_{11}^{**}(\beta)}{E^{-1} + R_{11}^*(\beta)}, \quad \eta = \frac{R_2^{**}(\beta)}{-2E^{-1} + R_2^*(\beta)} - \frac{R_{11}^{**}(\beta)}{E^{-1} + R_{11}^*(\beta)} \\ R_{ij}^*(\beta) = \int_0^\infty K_{ij}(S)\cos\beta S\,dS, \quad R_{ij}^{**}(\beta) = \int_0^\infty K_{ij}(S)\sin\beta S\,dS \\ K_1(S) = -4E^{-1}[K_{11}(S) + 2K_{12}(S) + K_{22}(S) - K_{33}(S)] - \\ 2[K_{33}^{22}(S) - K_{22}^{22}(S) + K_{12}^{33}(S) + 2K_{12}^{11}(S)] + \\ K_{22}^{22}(S) + K_{33}^{33}(S) + K_{12}^{12}(S) \\ K_2(S) = K_{12}(S) + K_{22}(S) - K_{33}(S) \end{gathered} \tag{2.6}$$

where $K_{12}^{11}(S)$, for instance, has the following construction:

$$K_{12}^{11}(S) = \int_0^S K_{11}(S)K_{12}(S - s)\,ds$$

Taking account of (2.4)—(2.6) and extracting the real part, we obtain an expression for the stress function from (2.2)

$$\begin{gathered} \varphi(x, y) = \int_0^\infty P_1(\beta, y)\cos\beta x\,d\beta - \int_0^\infty P_2(\beta, y)\sin\beta x\,d\beta \\ \begin{Bmatrix} P_1(\beta, y) \\ P_2(\beta, y) \end{Bmatrix} = \frac{B_+B_-}{M}\left\{\exp(-T_1 y)\left[\frac{B_+}{B_-}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} T_2 y - \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(T_2 y - T_5)\right] + \right. \\ \left. \exp(-T_3 y)\left[\frac{B_-}{B_+}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} T_4 y - \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(T_4 y + T_5)\right]\right\} \\ M = B_+^2 + B_-^2 - 2B_+B_-\cos T_5, \quad T_5 = {}^1/_2(A_{1+}A_{2+}^{-1} - A_{1-}A_{2-}^{-1}) \\ \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = B_-\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}({}^1/_2 A_{1-}A_{2-}^{-1}), \quad \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = B_+\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}({}^1/_2 A_{1+}A_{2+}^{-1}) \end{gathered}$$

By knowing the stress function $\varphi(x, y)$ the stress components $\sigma_x, \sigma_y, \tau_{xy}$ can be determined; then by using the relationship between the strain components and the stress components for a viscoelastic body, we find $\varepsilon_x, \varepsilon_y, \gamma_{xy}$, and finally, determine the displacements u and v. Thus

$$v(x)|_{y=0} = \int_0^\infty P_3(\beta)\cos\beta x\, d\beta + \int_0^\infty P_4(\beta)\sin\beta x\, d\beta \tag{2.7}$$

$$P_3(\beta) = [\nu E^{-1} + R_{22}^*(\beta)]\, Q_1(\beta) - R_{22}^{**}(\beta)\, Q_2(\beta) -$$

$$[E^{-1} + R_{12}^*(\beta)]\, Q_3(\beta) + R_{12}^{**}(\beta)\, Q_4(\beta)$$

$$P_4(\beta) = R_{22}^{**}(\beta)\, Q_1(\beta) + [\nu E^{-1} + R_{22}^*(\beta)]\, Q_2(\beta) - R_{12}^{**}(\beta)\, Q_3(\beta) - [E^{-1} + R_{12}^*(\beta)]\, Q_4(\beta)$$

$$Q_l(\beta) = \int_0^\infty q_l(\beta, y)\, dy, \qquad l = 1, \ldots, 4$$

The polynomials $q_l(\beta, y)$ are sufficiently awkward in the general case. Let us present the expression for one of them

$$q_1(\beta, y) = \frac{B_+ B_-}{M}\Big\{\exp(-T_1 y)\Big[\frac{B_+}{B_-}(2T_1T_2\sin T_2 y +$$

$$(T_1^2 - T_2^2)\cos T_2 y) - (T_1^2 - T_2^2)\cos(T_2 y - T_5) -$$

$$2T_1T_2\sin(T_2 y - T_5)\Big] + \exp(-T_3 y)\Big[\frac{B_-}{B_+}(2T_3T_4\sin T_4 y +$$

$$(T_3^2 - T_4^2)\cos T_4 y) - (T_3^2 - T_4^2)\cos(T_4 y + T_5) -$$

$$2T_3T_4\sin(T_4 y + T_5)\Big]\Big\}$$

In order to extract the singular component out of the Green's function later, let us present the solution of the problem of a moving stamp acting on the boundary of and elastic half-plane in the form of a Fourier integral. The stress function will have the following form

$$\varphi^*(x, y) = -\int_0^\infty \exp(-\beta y)\,\beta^{-2}(1 + \beta y)\cos\beta x\, d\beta \tag{2.8}$$

The displacement component $v^*(x)$ can be represented thus:

$$v^*(x)|_{y=0} = -E^{-1}\int_0^\infty\int_0^\infty [(1-\nu) + (1+\nu)\beta y] \times \exp(-\beta y)\cos\beta x\, dy\, d\beta \tag{2.9}$$

3. Transformation of the Fourier integrals.

We consider the difference

$$I(x) = v(x)|_{y=0} - C^\circ v^*(x)|_{y=0}$$

$$C^\circ = \lim_{x\to 0}\Big\{\int_0^\infty P_3(\beta)\cos\beta x\, d\beta\Big[\int_0^\infty \frac{\cos\beta x}{\beta}\, d\beta\Big]^{-1}\Big\}$$

Taking account of (2.7) and (2.9), we will have

$$I(x) = \int_0^\infty \Big\{C^\circ E^{-1}\int_0^\infty [(1-\nu) + (1+\nu)\beta y]\exp(-\beta y)\, dy + P_3(\beta)\Big\}\cos\beta x\, d\beta + \int_0^\infty P_4(\beta)\sin\beta x\, d\beta \tag{3.1}$$

The integrands in (3.1) enclosed in the braces are regular functions. Denoting the first integral in (3.1) by $I_1(x)$ and the second by $I_2(x)$, we represent them as series of Chebyshev polynomials of the second kind $U_n(x)$ ($T_n(x)$ are Chebyshev polynomials of the first kind)

$$I_1(x) = \sum_{n=1}^\infty a_n U_n(x), \qquad I_2(x) = \sum_{n=1}^\infty b_n U_n(x), \quad a_n = \frac{2}{\pi}\int_{-1}^1 I_1(x)\, T_n(x)(1 - x^2)^{-1/2}\, dx,$$

$$b_n = \frac{2}{\pi}\int_{-1}^1 I_2(x)\, T_n(x)(1 - x^2)^{-1/2}\, dx, \quad (T_n(x) = \cos n \arccos x, \quad U_n(x) = \sin n \arccos x)$$

We use the notation $\theta = \arccos x$. Differentiating this equality with respect to x and taking into account that /2/

$$\int_0^\pi \cos(\beta\cos\theta)\cos n\theta\, d\theta = \pi\cos\frac{n\pi}{2}\, J_n(\beta)$$

where $J_n(\beta)$ is the Bessel function, we obtain the following relationships for the coefficients

a_n and b_n

$$a_n = 2\cos\frac{n\pi}{2}\int_0^\infty \left\{C^\circ E^{-1}\int_0^\infty [(1-\nu)+(1+\nu)\beta y]\exp(-\beta y)\,dy + P_3(\beta)\right\} J_n(\beta)\,d\beta$$

$$b_n = 2\sin\frac{n\pi}{2}\int_0^\infty P_4(\beta)\,J_n(\beta)\,d\beta$$

We finally have for the displacements $v(x)\,|_{y=0}$

$$v(x)\,|_{y=0} = -\frac{2C^\circ\nu}{E}\int_0^\infty \frac{\cos\beta x}{\beta}\,d\beta + \sum_{n=1}^\infty 2\cos\frac{n\pi}{2}\sin n\arccos x\times \tag{3.2}$$

$$\int_0^\infty\left[\frac{2C^\circ\nu}{E\beta}+P_3(\beta)\right]J_n(\beta)\,d\beta + \sum_{n=1}^\infty 2\sin\frac{n\pi}{2}\sin n\arccos x\int_0^\infty P_4(\beta)\,J_n(\beta)\,d\beta$$

4. Solution of the singular integral equation.

The question of determining the contact stresses under a stamp moving at a certain constant velocity along the boundary of a viscoelastic half-plane can be reduced to the solution of the singular integral equation

$$L\frac{\partial v}{\partial \bar{x}}\Big|_{y=0} = \int_{-1}^{1} P(\bar{\xi})\,K(\bar{x}-\bar{\xi})\,d\bar{\xi} \tag{4.1}$$

$$(L = E\,/\,(2C^\circ\nu),\quad P(\bar{\xi}) = \sigma_y(\bar{\xi})\,|_{y=0},\quad \bar{x} = x\,/\,a,\quad \bar{\xi} = \xi\,/\,a)$$

Here $\bar{x}, \bar{\xi}$ are new independent variables (we later omit the upper bar), $|\,x\,| \leqslant a$ is the contact area which is considered given in the case under consideration. The kernel of the integral equation (4.1) has the form

$$K(x-\xi) = \mathrm{Re}\int_0^\infty \exp[i\beta(x-\xi)]\left[\frac{k_4}{k_4-k_3}\exp(k_3 y) + \frac{k_3}{k_3-k_4}\exp(k_4 y)\right]\Big|_{y=0} d\beta = \frac{1}{\xi - x} + K^*(x-\xi)$$

Since $K^*(x-\xi)$ is a regular function, we represent it as the sum of a product of polynomials $P_n(x)$ by the Chebyshev polynomials $T_n(\xi)$:

$$K^*(x-\xi) = \sum_{n=0}^{k} P_n(x)\,T_n(\xi)$$

To determine $P_n(x)$ we use the property of orthogonality of the Chebyshev polynomials, then

$$P_m(x) = \int_{-1}^{1} K^*(x-\xi)\,T_m(\xi)\,(1-\xi^2)^{-1/2}\,d\xi$$

We seek the function characterizing the contact stress $P(\xi)$ in the form of the series

$$P(\xi) = \sum_{n=0}^{\infty} A_n T_n(\xi)\,(1-\xi^2)^{-1/2}$$

We will thus have

$$L\frac{\partial v}{\partial x}\Big|_{y=0} = \sum_{n=0}^\infty A_n\int_{-1}^{1}\frac{T_n(\xi)}{(1-\xi^2)^{1/2}}\,\frac{d\xi}{\xi - x} + \sum_{n=0}^\infty A_n\int_{-1}^{1}\frac{T_n(\xi)}{(1-\xi^2)^{1/2}}\sum_{m=0}^{k} P_m(x)\,T_m(\xi)\,d\xi \tag{4.2}$$

Let us transform the integral under the first summation sign in (4.2). We introduce the notation $x = \cos\varphi$ and then according to /2/

$$\int_{-1}^{1}\frac{T_n(\xi)}{(1-\xi^2)^{1/2}}\,\frac{d\xi}{\xi - x} = \pi\frac{\sin n\varphi}{\sin\varphi} = \frac{\pi}{n}\,T_n{}'(x)$$

Taking into account the orthogonality of the Chebyshev polynomials, we will have

$$L\frac{\partial v}{\partial x}\Big|_{y=0} = \sum_{n=0}^\infty A_n\frac{\pi}{n}\,T_n{}'(x) + \sum_{n=1}^{k} A_n\frac{\pi}{2}\,P_n(x) + A_0\pi P_0(x)$$

If the quantity $(\partial v/\partial x)|_{y=0}$ can be expanded in derivatives of the Chebyshev polynomials

$$L\frac{\partial v}{\partial x}\Big|_{y=0} = \sum_{n=0}^{k} B_n T_n'(x) + \sum_{n=k+1}^{\infty} B_n T_n'(x)$$

then we obtain two systems of algebraic equations to find the coefficients A_n

$$\sum_{n=0}^{k} B_n T_n'(x) = \sum_{n=0}^{k} A_n \frac{\pi}{n} T_n'(x) + \sum_{n=1}^{k} A_n \frac{\pi}{2} P_n(x) + A_0 \pi P_0(x) \tag{4.3}$$
$$\sum_{n=k+1}^{\infty} B_n T_n'(x) = \sum_{n=k+1}^{\infty} A_n \frac{\pi}{n} T_n'(x)$$

Equating terms in identical powers of x, we obtain k equations with k unknowns A_n. For $n > k$ the remaining coefficients A_n are found from the relationship $A_n = B_n\, n/\pi$.

5. Example. To illustrate the general method of solving the problem, we examine the particular case when only shear creep holds. It has been shown /3/ that on the basis of numerous experimental investigations materials of the type AG-4S or SVAM can be considered as an elastically orthotropic body with shear creep. This permits writing the relationships between the strain and stress components in the following form

$$\varepsilon_x = E^{-1}\sigma_x - \nu E^{-1}\sigma_y, \quad \varepsilon_y = E^{-1}\sigma_y - \nu E^{-1}\sigma_x$$
$$\gamma_{xy} = 2(1+\nu) E^{-1}\tau_{xy} + \int_{-\infty}^{t} \tau_{xy} K^*(t-\tau)\, d\tau$$

Using the deductions obtained above, we represent the expression for the stress function in the form

$$\varphi(x, y) = \int_0^\infty P_1(\beta, y)\cos\beta x\, d\beta + \int_0^\infty P_2(\beta, y)\sin\beta x\, d\beta$$

where the polynomials $P_1(\beta, y)$ and $P_2(\beta, y)$ have the form indicated in Sect.2, and according to the conditions presented above, the quantities therein are

$$D = {}^1/_2 E\beta^2 (\chi_1^2 + \chi_2^2)^{1/4}, \quad F = {}^1/_2 E\beta^3 (\chi_3^2 + \chi_4^2)^{1/2}; \quad \alpha = \chi_1/\chi_2, \quad \eta = -\chi_3/\chi_4$$
$$\begin{Bmatrix}\chi_1\\ \chi_2\end{Bmatrix} = \int_0^\infty \left[4E^{-1} + \int_0^S K(S-s)\, ds\right] K(S) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} \beta S\, dS$$
$$\chi_3 = \int_0^\infty K(S)\sin\beta S\, dS, \quad \chi_4 = 2E^{-1} + \int_0^\infty K(S)\cos\beta S\, dS$$

By having a specific expression for the kernel $K(S)$ and performing the computations presented in the paper in sequence, we find: $\sigma_x, \sigma_y, \tau_{xy}, \varepsilon_x, \varepsilon_y, \gamma_{xy}, u, v$.

Remark. Existing test data /1/ permit the assertion that the volume strain is purely elastic in many cases, and there is no volume aftereffect. We require the volume compression operator to be constant. Moreover, by considering the viscoelastic body isotropic, we obtain $K_{11}^* = K_{22}^* = K_{12}^* = {}^1/_4 K_{33}^* = K^*$.

Taking all the above into account, we write the relation between the strain and stress components. According to (2.1), we form an equation for the stress function. By seeking the solution in the form (2.2), we arrive at a differential equation of the form (2.3). The roots of the corresponding characteristic equation are $k_3 = k_4 = -\beta$, i.e., the problem is reduced to the corresponding problem for an elastic half-plane.

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